RA 14

8.3 Summations

- **Summation Notation** is used to express the sum of the terms in a numerical sequence.

- $\sum_{i=s}^{t} a_i = a_s + a_{s+1} + \dots + a_t$

    - The variable i is called the index.
    - The value s is the lower limit
    - The value t is the upper limit

- $\sum_{j=1}^{n} j+1 = \left(\sum_{j=1}^{n} j\right) + 1$

- Pulling out a final term from a summation

$$\sum_{K=m}^{n} a_K = \left(\sum_{K=m}^{n-1} a_K\right) + a_n$$

$$\sum_{j=1}^{n} (j+1)^2 = \underbrace{(1+1)^2 + (2+1)^2 + \dots + \boxed{(n-1)+1}^2}_{\sum_{j=1}^{n-1} (j+1)^2} + (n+1)^2$$

$$\sum_{j=1}^{n-1} (j+1)^2 + (n+1)^2$$

Change of variables example

Original : $\sum_{k=1}^{n} (k+2)^3$

using $i = k+2$: $\sum_{3}^{n+2} i^3$

- A <u>closed form</u> for a sum is a mathematical expression that expresses the value of the sum without summation notation.

8.8 Recursive Definitions

- In a <u>recursive definition</u> of a function, the value of the function is defined in terms of the output value of the function on smaller inputs values.
  - Recursive definition for factorial:

    $n! = f(n)$ such that:

    $f(0) = 1$

    $f(n) = n \cdot f(n-1)$ for $n \geq 1$

- <u>Recursion</u> is the process of computing the values of a function using the result of the function on smaller input values.

Recursively defined sets

Components of a recursive definition of a set:

- A basis explicitly states that one or more specific elements are in a set
- A recursive rule shows how to construct additional elements in the set from elements already known to be in the set. (There is often more than one recursive rule)
- An exclusion statement states that an element is in the set only if it is given in the basis or can be constructed by applying the recursive rules repeatedly to elements given in the basis.

- Binary Strings
  - The set B^k, where $B = \{0, 1\}$ is defined to be the set of all binary strings of length k.
  - B^* → the set of all binary strings without any restriction on length, this is an infinite set.
  - λ → the empty string whose length is 0.

- Recursive definition of B^* is:
  - Base case $\lambda \in B^*$
  - Recursive rule: if $x \in B^*$ then,
    - $x0 \in B^*$
    - $x1 \in B^*$

- Recursive definition for perfect binary trees:
  - A tree has vertices and edges that connect pairs of vertices.
  - Each perfect binary tree has a designated vertex called the root.

- Recursive definition for the set of perfect binary trees:
  - Basis: A single vertex with no edges is a perfect binary tree. The root is the only vertex.
  - Recursive Rule: If T is a perfect binary tree, then a new perfect binary tree T' can be constructed by taking two copies of T, adding a new vertex v and adding edges between v and the roots of each copy of T. The new vertex v is the root of T'.

8.9 Structural induction

- Structural Induction is a type of induction used to prove theorems about recursively defined sets that follows the structure of the recursive definition.

- A string of parantheses is balanced if the number of left parantheses is equal to the number of right parentheses.

8.10 Recursive Algorithms

- A recursive algorithms is an algorithm that calls itself.

- An algorithm's calls to itself are known as recursive calls.